Prob 140 Spring 2018 Final Exam

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1. Let $X_1, X_2, \ldots$ be independent and identically distributed with the distribution given in the table below. For each $n \geq 1$ let $S_n = X_1 + X_2 + \ldots + X_n$.

value	-1	0	1
probability	$1/8$	$6/8$	$1/8$

a) Find the decimal value of $SD(X_1)$.
b) Find $P(S_2 > 0)$. Leave the calculation unsimplified.
c) Find an approximate decimal value of $P(S_{400} > 20)$.

2. A random vector $\mathbf{R}$ has the multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The components are as follows:

$$\mathbf{R} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_U \\ \mu_V \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 4.8 & 1.4 \\ 4.8 & 9 & 0 \\ 1.4 & 0 & 1 \end{bmatrix}$$

a) Find $E(VW)$.
b) Find $E(U \mid V)$.
c) Find $Var(U \mid V)$.
d) Let b be a constant. What is the joint distribution of U and $V - bW$?

3. In the problems below, do not leave integrals or infinite sums in your answer.

a) Let X have the Poisson (λ) distribution. Let Y be independent of X and let Y have the geometric (p) distribution on $\{0, 1, 2, \ldots\}$. That is, let Y be the number of failures before the first success in i.i.d. Bernoulli (p) trials. Find $P(X = Y)$.

b) Let U have the uniform distribution on the interval $(0, 1)$. Given $U = u$, let V have the geometric (u) distribution on $\{0, 1, 2, \ldots\}$. Find the survival function of V.

4. The code below consists of a sequence of assignment statements. You can assume that all the necessary libraries have been imported.

```
iid_sample = stats.uniform.rvs(0, 1, size=10)
sorted_sample = np.sort(iid_sample)              # sorts in increasing order
p = sorted_sample.item(6)
h = stats.binom.rvs(100, p, size=1)
y = stats.binom.rvs(10000, p, size=1) / 10000
```

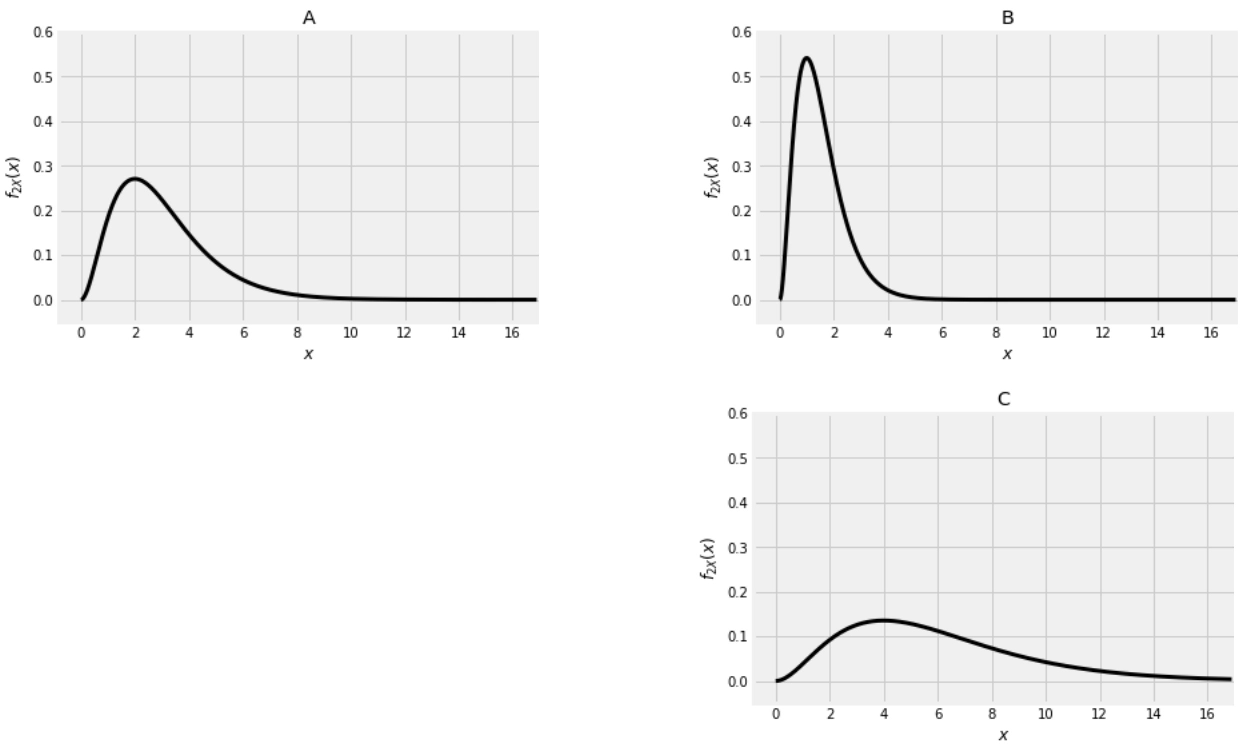
a) After the code is run, `p` is the observed value of a random variable X. What is the unconditional distribution of X? Explain or calculate.

b) Also, `h` is the observed value of a random variable H. What is the conditional distribution of X given $H = h$? Explain or calculate.

c) Finally, `y` is the observed value of a random variable Y. What is the approximate unconditional distribution of Y? Explain or calculate.

5. Let X have the gamma $(3, 2)$ distribution and let Y independent of X have the gamma $(2, 3)$ distribution.

a) Let f_{2X} be the density of $2X$. One of the following is the graph of f_{2X}. Which one? Circle the letter above your chosen figure and explain your choice.



b) What is the density of $2X + 3Y$? Explain or calculate.

c) Write a formula for $P(2X > 3Y)$. You can leave any integrals and algebra unsimplified.

6. I have n balls of crumpled paper that I am trying to throw into a garbage can in a corner of my office.

- At the first stage, I throw the n balls one by one. Assume that each ball falls into the can with probability p_1, independently of all the others. Let X_1 be the number of balls that fall into the garbage can at this stage.
- At the second stage, I pick up all the balls that did not fall in the garbage can at the first stage, and throw them again. Assume that at this stage each ball falls into the can with probability p_2, independently of all the others. Let X_2 be the number of balls that fall into the garbage can at this stage. Note that if $X_1 = n$ then $X_2 = 0$.

a) Find $E(X_2 \mid X_1)$.

b) Find $E(X_2)$.

c) Find $Var(X_2)$.

d) Find the distribution of $X_1 + X_2$. You should not have to do a lot of work for this one. Think about what the variable represents, and start with the possible values.

7. Let X have the Cauchy density.

a) Find the density of $1/X$. Recognize it as one of the famous ones and say which one it is. The answer might surprise you.

b) Let $V = 1/(1 + X^2)$. Find the density of V. Recognize it as one of the famous ones and provide its name and parameters.

8. A bowl contains n pieces of cooked spaghetti, each of which has two ends.

- At Stage 1, pick two of the $2n$ ends at random without replacement and tie them together.

• At Stage 2, pick two of the remaining untied ends at random without replacement and tie them together.

• Continue in this way until there are no untied ends left in the bowl.

a) What is the chance that you form a loop at Stage 1? Simplify your answer as much as possible, as this will help in a later part.

b) How many untied ends remain in the bowl after Stage 1 is completed?

c) Find the expected number of loops when there are no untied ends left in the bowl.

d) Recall from one of the labs that $\sum_{j=1}^{m} \frac{1}{j} \approx \log(m)$ for large m. Show that for large n, the expected number of loops is approximately $\log(2\sqrt{n})$. Start by splitting $\sum_{j=1}^{m} \frac{1}{j}$ appropriately into two pieces.

9. Let X have the Poisson distribution with expectation 1. Prove the following tail bound:

For every $k > 0$,

$$P(X \geq k) \;\leq\; \left(\frac{e}{k}\right)^k$$