Prob 140 Spring 2017 Final Exam

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1. A fair die with two red faces and four green faces is rolled repeatedly.

a) Find the chance that both colors appear among the first 12 rolls.

b) Find the expected number of rolls needed for the color green to appear a total of 15 times.

c) Find the chance that the color green appears more often than the color red among the first 10 rolls.

d) Given that the color green appeared 9 times among the first 14 rolls, what is the chance that green did not appear among the first three rolls?

2. Let V and W have joint density given by

$$f(v,w) = \begin{cases} 2e^{-v-w}, & 0 < v < w < \infty \\ 0 & \text{otherwise} \end{cases}$$

a) Find the density of V.

b) Find the survival function of $W - V$.

c) Find $E(W)$.

3. Let N, n, and m be positive integers with $N = n + m$. Suppose a list of N numbers has mean μ and variance σ^2. Let $\{X_1, X_2, \ldots, X_n\}$ be a set of n numbers drawn at random **without** replacement from this list, and let $\{Y_1, Y_2, \ldots, Y_m\}$ be the set of $m = N - n$ numbers that are not drawn. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \bar{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$$

a) Fill in the blanks: $E(\bar{X}) =$ ____________________ and $SD(\bar{X}) =$ ____________________. You don't have to derive the answers if you simply remember what they are.

b) Write μ in terms of $\bar{X}$, $\bar{Y}$, n, and m.

c) Let $D = \bar{X} - \bar{Y}$. Notice that D is a test statistic you used when you were performing permutation tests in Data 8 (but you don't have to know that to do this problem). Use parts (a) and (b) to fill in the blanks:

$E(D) =$ ________________________ $SD(D) =$ ________________________

4. Let X and Y be independent random variables. Let X have moment generating function

$$M_X(t) = e^{5t+2t^2}, \qquad -\infty < t < \infty$$

and let Y have moment generating function

$$M_Y(t) = e^{8t^2}, \qquad -\infty < t < \infty$$

a) Find the moment generating function of $X - 2Y - 3$.

b) Find $P(X > 2Y + 3)$.

5. Let $\theta > 0$ be a constant, and let X have the beta $(1, \theta)$ density.

a) Find the density of $-\log(1 - X)$. Recognize it as one of the famous ones and give its name and parameters.

b) Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from the beta $(1, \theta)$ distribution. Find the maximum likelihood estimate of θ.

6. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with the normal (μ, σ^2) distribution. Define the sample mean M as

$$M = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and for each i in the range 1 through n let the ith deviation from mean be D_i defined by

$$D_i = X_i - M$$

a) Find the joint distribution of D_1 and D_2.
b) Pick one option and justify your choice.
The random variables M and D_1 are
(i) neither uncorrelated nor independent.
(ii) uncorrelated but not independent.
(iii) independent but not uncorrelated.
(iv) uncorrelated and independent.

7. A data scientist draws a bootstrap sample from an original random sample of n individuals. Recall from Data 8 that the bootstrap sample consists of n draws made at random with replacement from the n indviduals in the original sample.

Let N be the number of individuals in the original sample who don't appear in the bootstrap sample.
a) Find $E(N)$.
b) Find $Var(N)$.

8. Let M have the gamma (r, λ) density. Given $M = m$, let N have the Poisson distribution with parameter m. In each part below, fill in the blanks.
a) $E(N \mid M) =$ ________________________________
b) $Var(N \mid M) =$ ________________________________
c) $E(N) =$ ________________________________
d) $Var(N) =$ ________________________________
e) For $m > 0$ and non-negative integer n,
$P(M \in dm, N = n) \sim$ ____
f) The posterior distribution of M given $N = n$ is ___________________________

(give the name of a standard distribution) with parameters ____.

9. Let Z have the standard normal density. Then $E(Z^k)$ is well defined and finite for every positive integer k. In this question you will find the numerical value of $E(Z^k)$ for each positive integer k.

a) Let n be a positive integer and consider the odd integer $k = 2n - 1$. What is the value of $E(Z^{2n-1})$, and why?

b) Write the formula for the density of Z^2. You don't have to derive the formula if you remember it or can work it out from the formula sheets.

c) Let n be a positive integer and consider the even integer $k = 2n$. Use part (b) to find $E(Z^{2n})$ in terms of the Gamma function.

d) For each positive integer n, find an integer c_n such that $E(Z^{2n}) = c_n E(Z^{2n-2})$. Then use induction to derive a formula for $E(Z^{2n})$ that does not involve the Gamma function.