Prob 140, Berkeley, Spring 2018
MATH PREREQUISITES
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Here are some algebra and calculus exercises that you should be able to do with ease. I expect you to know with complete confidence that you have worked these correctly, just as you would know with complete confidence that $2 + 3 = 5$.

Some of you will find this straightforward, others will have to dredge up the relevant math from the depths of memory, and most will be somewhere in between. Regardless of where you fall on that spectrum, please make sure that you thoroughly understand the math on these pages. It's at the level that I will take for granted in lecutre.

Reference: Appendices at the back of Pitman's text have summaries of useful results.

FINITE SUMS

1. Consider the sequence defined by $c_i = i$, for $i = 1, 2, \dots, 10$.

a) If possible, find $\sum_{k=1}^{10} c_k$. If this is not possible, explain why not.

b) Prove a simple formula for $\sum_{i=1}^{n} i$.

2. Does the expression $\sum_{n=1}^{10} 2$ make sense? If it does, what is its value?

3. Let $\{c\}$ and $\{d\}$ be sequences of real numbers so that

$$\sum_{i=1}^{100} c_i \;=\; 10 \quad \text{and} \quad \sum_{j=1}^{100} d_j \;=\; 20$$

In parts a)-c) find the value of the expression.

a) $\sum_{i=1}^{100}(4c_i + 5)$

b) $\sum_{i=1}^{100} 4c_i \;+\; 5$

c) $\sum_{i=1}^{100}(4c_i - d_i + 5)$

d) True or false:

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(c_i + d_j) \;=\; \sum_{i=1}^{100}(c_i + d_i)$$

If the identity is true, find the common value of the two sides. If it is false, can you find the values of the two sides?

4. Fill in the blanks.

$$\sum_{i=1}^{n}\sum_{j=i}^{n} a_{ij} \;=\; \sum_{j=\_\_}^{\_\_}\;\sum_{i=\_\_}^{\_\_} a_{ij}$$

INFINITE SUMS

5. Let $0 < p < 1$. Find simple expressions for

a) $\sum_{i=0}^{100} p^i$ **b)** $\sum_{i=0}^{\infty} p^i$ **c)** $\sum_{i=100}^{\infty} p^i$

6. The sum $\sum_{n=0}^{\infty} 1/n!$ can be expressed very simply. Find that simple expression and a numerical value.

7. Repeat the previous exercise for each of the sums $\sum_{i=0}^{\infty} 2^i/i!$ and $\sum_{i=0}^{\infty} 2^{3i}/i!$. If you had trouble with the previous exercise, this one might help.

8. Find $\sum_{i=0}^{\infty} 2^i/(i+1)!$

EXPONENTIAL APPROXIMATIONS

You know that $e^0 = 1$. What we're going to need, quite often, is an approximation to e^x for a small non-zero number x. A crude approximation is 1 because x is tiny. But you can get a finer approximation by writing the first two terms in the expansion for e^x and remembering that Taylor says the rest is small compared to x.

9. Explain why $e^{0.01}$ is roughly 1.01 and $e^{-0.01}$ is roughly 0.99.

10. Use your reasoning in the previous exercise to explain why $\log(1+x)$ is roughly x for small x. In this class, as in much of math, log is taken to the base e.

11. Suppose $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \mu$ for some positive number μ. Use the previous exercise to find $\lim_{n\to\infty}(1-p_n)^n$. Start by considering $\log(1-p_n)^n$.

COMBINATORICS

12. How many different ways are there to arrange six people

a) in a row? **b)** in a circle?

13. A committee consists of 6 women and 4 men. How many different choices can be made if you want to select

a) a Chairperson and an Assistant Chairperson?

b) a subcommittee of two people?

c) a committee of two men and two women?

14. Let i and j be integers such that $1 \le i, j \le n$. How many pairs (i, j) can be formed such that

a) $i \neq j$? **b)** $i < j$?

15. Let a and b be any two real numbers. You know that $(a+b)^2 = a^2 + 2ab + b^2$.

a) Analogously, write the following as a sum of four terms: $(a+b)^3$

b) Let n be a non-negative integer. Fill in the blanks:

$$(a+b)^n \;=\; \sum_{k=\_}^{\_} \_\_\_ a^k b^{n-k}$$

CALCULUS

16. Calculate the following.

a) $\frac{d}{dx}\log(x^2)$

b) $\frac{d}{dx}xe^{-cx}$ where $c > 0$ is a constant

c) $\int xe^{-cx}dx$ where $c > 0$ is a constant (use part (b) or methods of integration)

d) $\int_0^\infty ce^{-cx}dx$ where $c > 0$ is a constant

17. Let $c > 0$ be a constant. $\int_0^x ce^{-cx}dx$ doesn't make sense. Why not?

18. Let r be a positive number. The number $\Gamma(r)$ is read as "gamma of r" and is defined by

$$\Gamma(r) \;=\; \int_0^\infty t^{r-1}e^{-t}dt$$

You can assume that the integral converges and thus that $\Gamma(r)$ is well defined. If you don't know what that means, just assume there are no problems with the definition.

a) Show that $\Gamma(r+1) = r\Gamma(r)$. Use integration by parts.

b) Use induction and the previous part to show that if r is an integer then $\Gamma(r) = (r-1)!$.

c) Let λ be a positive number. Find

$$\int_0^\infty t^{r-1}e^{-\lambda t}dt$$

in terms of λ, r, and $\Gamma(r)$.

19. Calculate $\int_0^1\int_0^1(x + xy + y)dxdy$.

20. Fill in the blanks (it really helps to draw the region of integration):

$$\int_0^1\int_y^1(x + xy + y)dxdy \;=\; \int_0^1\int_{\_}^{\_}(x + xy + y)dydx$$